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LETTER TO THE EDITOR

Twist deformation of the rank-one Lie superalgebra

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Abstract. The Drinfeld twist is applied to deform the rank-one orthosymplectic Lie superalgebra $osp(1|2)$. The twist element is the same as for the $sl(2)$ Lie algebra due to the embedding of the $sl(2)$ into the superalgebra $osp(1|2)$. The R-matrix has the direct sum structure in the irreducible representations of $osp(1|2)$. The dual quantum group is defined using the FRT-formalism. It includes the Jordanian quantum group $SL_\xi(2)$ as subalgebra and Grassmann generators as well.

1. The deformed algebra $osp_\xi(1|2)$

It is difficult to overestimate the role of the rank-one Lie algebra $sl(2)$ in the theory of Lie groups and their applications. The corresponding role for Lie superalgebras is played by the orthosymplectic superalgebra $osp(1|2)$ with five generators $\{h, X_-, X_+, v_-, v_+\}$ and commutation relations (Lie super- or $\mathbb{Z}_2$ graded-brackets):

$$[h, X_\pm] = \pm 2X_\pm \qquad [X_+, X_-] = h \tag{1}$$

$$[h, v_\pm] = \pm v_\pm \qquad [v_+, v_-]_+ = -h/4 \tag{2}$$

$$[X_\pm, v_\pm] = 0 \qquad [X_\pm, v_\mp] = v_\pm \qquad [v_\pm, v_\pm]_+ = \pm X_\pm/2. \tag{3}$$

The generators h and $X_\pm$ are even (zero parity, $p = 0$), while $v_\pm$ are odd, $p = 1$. As a Hopf superalgebra, the universal enveloping $\mathcal{U}\big(osp(1|2)\big)$ of $osp(1|2)$ is generated, as $sl(2)$, just by three elements: it is sufficient to start from $\{h, v_-, v_+\}$ restricted by the relations (2) only, and define $X_\pm \equiv \pm 4v_\pm^2$.

The quantum deformation of $sl(2)$ can be considered as a 'pivot' of the quantum group theory [1, 2], while the corresponding quantum superalgebra $osp_q(1|2)$ constructed in [3–5], is the corresponding analogue for the quantum supergroups. As a quasitriangular Hopf superalgebra $osp_q(1|2)$, analogously to the universal enveloping of $osp(1|2)$, is generated by three elements $\{h, v_-, v_+\}$ under the relations

$$[h, v_\pm] = \pm v_\pm \qquad [v_+, v_-] = -\tfrac{1}{4}(q^h - q^{-h})/(q - q^{-1}).$$

It is worth noting that, while $sl(2)$ is embedded into $osp(1|2)$, such embedding does not exist for $sl_q(2)$ into $osp_q(1|2)$ because the coproduct of even elements $X_\pm \sim v_\pm^2$ also includes odd ones.

The aim of this paper is to construct and study the twist deformation [6] of $osp(1|2)$ that looks, in some sense, more natural than $osp_q(1|2)$ because it is consistent with this

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fundamental property of inclusion $sl(2) \subset osp(1|2)$ and it is generated by the same twist element of $sl(2)$.

The triangular Hopf algebra $sl_\xi(2)$ (cf [7–12] and references therein) is given by the extension of the twist deformation of the universal enveloping of the Borel subalgebra $B_- \equiv \{h, X_-\}$ to the whole $\mathcal{U}(sl(2))$. The twist element $\mathcal{F}$ is

$$\mathcal{F} = 1 + \xi h \otimes X_- + \frac{\xi^2}{2} h(h+2) \otimes X_-^2 + \cdots$$

that can be written as

$$\mathcal{F} = (1 - 2\xi 1 \otimes X_-)^{-\frac{1}{2}(h\otimes 1)} = \exp(\tfrac{1}{2} h \otimes \sigma) \tag{4}$$

where $\sigma = -\ln(1 - 2\xi X_-)$.

Let us recall from [6] that for a quasitriangular Hopf algebra $\mathcal{A}$ with an R-matrix $\mathcal{R}$ the twisted Hopf algebra $\mathcal{A}_t$ has R-matrix $\mathcal{R}^{(\mathcal{F})}$ given by the twist transformation

$$\mathcal{R}^{(\mathcal{F})} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1} \tag{5}$$

of the original R-matrix $\mathcal{R}$, where $\mathcal{F}_{21} = \mathcal{P}\mathcal{F}\mathcal{P}$, and $\mathcal{P}$ is the permutation map in $\mathcal{A} \otimes \mathcal{A}$. The algebraic sector of $\mathcal{A}_t$ is not changed and the new coproduct is $\Delta_t = \mathcal{F}\Delta\mathcal{F}^{-1}$. The twist element satisfies the relations in $\mathcal{A} \otimes \mathcal{A}$ [6]

$$(\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = 1$$

and in $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta)\mathcal{F}.$$

According to this Drinfeld definition, the algebraic relations of equations (1) for the twisted $sl(2)$ are still the same, while the twisted coproduct $\Delta_t \equiv \mathcal{F}\Delta\mathcal{F}^{-1}$ is now on the generators

$$\begin{aligned}
\Delta_t(h) &= h \otimes \mathrm{e}^{\sigma} + 1 \otimes h \\
\Delta_t(X_-) &= X_- \otimes 1 + 1 \otimes X_- - 2\xi X_- \otimes X_- = X_- \otimes \mathrm{e}^{-\sigma} + 1 \otimes X_- \\
\Delta_t(X_+) &= X_+ \otimes \mathrm{e}^{\sigma} + 1 \otimes X_+ - \xi h \otimes \mathrm{e}^{\sigma} h + \frac{\xi}{2} h(h-2) \otimes \mathrm{e}^{\sigma}(1 - \mathrm{e}^{\sigma}).
\end{aligned}$$

Let us stress that this twist of the whole $sl(2)$ is obtained due to the embedding $B_- \subset sl(2)$.

Thus, knowing that $B_- \subset sl(2) \subset osp(1|2)$, the procedure can be simply iterated to find $osp_\xi(1|2)$ (as well as the twisted deformations of all other nontrivial embeddings of B_-). It is an easy exercise, keeping in mind the expression of $\mathcal{F}$ (equation (4)), commutation relations (2) and (3), and the primitive coproduct of $osp(1|2)$, to obtain:

$$\begin{aligned}
\Delta_t(h) &= h \otimes \mathrm{e}^{\sigma} + 1 \otimes h \\
\Delta_t(v_-) &= v_- \otimes \mathrm{e}^{-\sigma/2} + 1 \otimes v_- \\
\Delta_t(v_+) &= v_+ \otimes \mathrm{e}^{\sigma/2} + 1 \otimes v_+ + \xi h \otimes v_- \mathrm{e}^{\sigma}.
\end{aligned} \tag{6}$$

One can reproduce the coproducts of $X_\pm$ by squaring the coproducts of $v_\pm$, taking into account the Z_2-grading of the tensor product:

$$(x \otimes y)(u \otimes w) = (-1)^{p(u)p(y)}(xu \otimes yw)$$

and the commutation relations (2) and (3).

The maps of co-unit ϵ and antipode S, necessary for a Hopf superalgebra definition, are

$$\begin{aligned}
&\epsilon(h) = \epsilon(v_\pm) = 0 \qquad \epsilon(1) = 1 \\
&S(h) = -h\mathrm{e}^{-\sigma} \qquad S(v_-) = -v_-\mathrm{e}^{\sigma/2} \qquad S(v_+) = -(v_+ - \xi h v_-)\mathrm{e}^{-\sigma/2}.
\end{aligned} \tag{7}$$

We can thus arrive at the following.

Definition. The Hopf superalgebra generated by three elements $\{h, v_-, v_+\}$ satisfying the relations (2), (6) and (7) is said to be the twist deformation of $\mathcal{U}(osp(1|2))$ or $osp_\xi(1|2)$.

This is a triangular Hopf superalgebra ($\mathcal{R}_{21}\mathcal{R} = 1$) with universal R-matrix

$$\mathcal{R} = \exp(\tfrac{1}{2}\sigma \otimes h) \exp(-\tfrac{1}{2}h \otimes \sigma). \tag{8}$$

The irreducible finite-dimensional representations of $osp_\xi(1|2)$

$$\rho_s : osp_\xi(1|2) \longrightarrow \mathrm{End}(W_s)$$

are the same as for $osp(1|2)$, due to the unchanged algebraic relations (2). They are parametrized by the half-integer spin $s = 0, \frac{1}{2}, 1, \ldots$, have dimension $4s + 1$, and are decomposed into a direct sum of two irreps of the $sl(2)$ [13]: $W_s = V_s + V_{s-\frac{1}{2}}$. Hence, the R-matrix in the irreps of $osp_\xi(1|2)$ is a direct sum of four R-matrices of $sl_\xi(2)$. For the first non-trivial case $s = \frac{1}{2}$ one obtains

$$\mathbf{R} = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}})\mathcal{R} = R(\xi) + I_2 + I_2 + 1 \tag{9}$$

where I_2 are 2×2 unit matrices, and $R(\xi)$ is the Jordanian solution to the Yang–Baxter equation (cf [7])

$$R(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix}. \tag{10}$$

The twist parameter can be scaled: $\xi \to \exp(2u)\xi$ by the similarity transformation with the element $\exp(-uh)$.

The basis of the irreps tensor product decomposition will include deformed Clebsch–Gordan coefficients, expressed as linear combinations of the usual ones and the matrix elements of the twist $\mathcal{F}$ [14]. This is reflected in the spectral decomposition of the R-matrix itself in the tensor product $W_s \otimes W_l$

$$\hat{R}^{s,l} = F^{s,l}\left(\sum_{j=|s-l|}^{s+l} (\pm) P^j\right)(F^{s,l})^{-1}$$

where P^j are projectors onto irreducible representations of $osp(1|2)$.

2. Quantum supergroup $OSp_\xi(1|2)$

The self-dual character of the twisted Borel subalgebra $(B_-)_\xi$ was pointed out in [8]. This is obvious in terms of the generators $\{h, \sigma\} \in (B_-)_\xi$ and the generators $\{s, p\} \in (B_-)'_\xi$ of the dual, with the only non-trivial evaluations $\langle h, s\rangle = 2$, $\langle \sigma, p\rangle = 2$ [8, 9]:

$$\begin{aligned}
&[h, \sigma] = 2(1 - e^\sigma) \qquad [p, s] = 2(1 - \mathrm{e}^s) \\
&\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma \qquad \Delta(s) = s \otimes 1 + 1 \otimes s \\
&\Delta(h) = h \otimes \mathrm{e}^\sigma + 1 \otimes h \qquad \Delta(p) = p \otimes \mathrm{e}^s + 1 \otimes p \\
&\epsilon(h) = \epsilon(\sigma) = 0 \qquad \epsilon(s) = \epsilon(p) = 0 \\
&S(h) = -h\mathrm{e}^{-\sigma} \qquad S(\sigma) = -\sigma \qquad S(p) = -p\mathrm{e}^{-s} \qquad S(s) = -s.
\end{aligned}$$

The situation is different for the twisted Hopf supersubalgebra $(sB_-)_\xi$. The latter is generated by two elements $\{h, v_-\}$ as $(B_-)_\xi$. However, due to the Z_2-grading its basis as a linear space consists of even $\sigma^m h^n$ and odd $\sigma^m v_- h^n$ elements $(\sigma = -\ln(1 + 8\xi v_-^2))$.

Proposition. The dual $(sB_-)'_\xi$ of the twisted Hopf superalgebra $(sB_-)_\xi$ is generated by three elements $\{\nu, \eta, x\}$ satisfying the relations

$$
\begin{aligned}
&[\nu, \eta] = 0 \qquad [\nu, x] = \tfrac{1}{2}(1 - \mathrm{e}^{-2\nu}) \qquad [x, \eta] = \tfrac{1}{2}\eta \qquad \eta^2 = 0 \\
&\Delta(\nu) = \nu \otimes 1 + 1 \otimes \nu \qquad \Delta(\eta) = \eta \otimes 1 + \mathrm{e}^{-\nu} \otimes \eta \\
&\Delta(x) = x \otimes 1 + \mathrm{e}^{-2\nu} \otimes x + \frac{1}{8\xi}\mathrm{e}^{-\nu}\eta \otimes \eta \\
&\epsilon(x) = \epsilon(\eta) = \epsilon(\nu) = 0 \\
&S(\eta) = -\eta \mathrm{e}^{\nu} \qquad S(\nu) = -\nu \qquad S(x) = -x\mathrm{e}^{2\nu}.
\end{aligned}
\tag{11}
$$

One can check this by a straightforward calculation of evaluating the dual basis $x^k \eta^\delta \nu^l$ of $(sB_-)'_\xi$ and $\sigma^m v_-^\delta h^n$ of $(sB_-)_\xi, k, l, m, n = 0, 1, 2, ...; \delta = 0, 1$ with the only non-zero evaluations among the generators: $\langle h, \nu\rangle = 1, \langle v_-, \eta\rangle = 1, \langle \sigma, x\rangle = 1$. We shall prove it below by a reduction from the quantum supergroup $OSp_\xi(1|2)$. The universal T-matrix (bicharacter) is given in term of these bases as a product of three exponents

$$\mathcal{T} = \exp(\sigma \otimes x) \exp(v_- \otimes \eta) \exp(h \otimes \nu).$$

It is interesting to point out that starting from a Hopf superalgebra without nilpotent elements we were forced to introduce Grassmannian variables (η) in the dual superalgebra.

The dual of the twisted Hopf superalgebra $osp_\xi(1|2)$ can be introduced using a Z_2-graded version of the FRT-formalism [2], because the R-matrix in the fundamental representation is known (9). The T-matrix of generators of quantum supergroup $OSp_\xi(1|2)$ in this representation has dimension 3×3. There are two convenient bases in this irrep as C^3: (i) with grading $(0, 1, 0)$ where the odd generators v_-, v_+ of $osp(1|2)$ are lower and upper triangular, and (ii) with grading $(0, 0, 1)$ where $\mathbf{T}$ can be written in block matrix form. Respectively, these forms are

$$\mathbf{T} = \begin{pmatrix} a & \alpha & b \\ \gamma & g & \beta \\ c & \delta & d \end{pmatrix} \qquad \mathbf{T} = \begin{pmatrix} T & \psi \\ \omega & g \end{pmatrix} \tag{12}$$

where $\mathbf{T}$ is 2×2 matrix of the even generators $\{a, b, c, d\}$, while ψ and ω are two component column $(\alpha, \delta)^t$ and row (γ, β) vectors of odd elements.

The 3×3 matrix $\mathbf{T}$ of the $OSp_\xi(1|2)$ generators satisfies the FRT-relation

$$\mathbf{R}\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1\mathbf{R} \tag{13}$$

with $\mathbb{Z}_2$-graded tensor product and 9×9 R-matrix $\mathbf{R}$ (9). From the block-diagonal form of $\mathbf{R}$ (9) it follows for 2×2 matrix T

$$R(\xi)T_1T_2 = T_2T_1R(\xi). \tag{14}$$

Hence, one reproduces the algebraic sector (commutation relations) of the twisted quantum group $SL_\xi(2)$ for the generators $\{a, b, c, d\}$ [7]. For the other blocks of different dimension we obtain from (13)

$$R(\xi)T_1\psi_2 = \psi_2T_1 \qquad g\mathbf{T} = \mathbf{T}g \tag{15}$$

$$\omega_1T_2 = T_2\omega_1R(\xi) \qquad \omega_1\psi_2 = -\psi_2\omega_1 \tag{16}$$

$$\omega_1\omega_2 = -\omega_2\omega_1R(\xi) \qquad R(\xi)\psi_1\psi_2 = -\psi_2\psi_1. \tag{17}$$

From the relations (14)–(17) one obtains centrality of the following elements:

$$\det_\xi T = a(d - \xi b) - cb \qquad g \qquad \theta = \omega T^{-1}\psi.$$

The coproduct, co-unit and antipode are given by the standard expressions of the FRT-formalism [2]

$$\Delta(\mathbf{T}) = \mathbf{T} \otimes \mathbf{T} \qquad \epsilon(\mathbf{T}) = I_3 \qquad S(\mathbf{T}) = \mathbf{T}^{-1}. \tag{18}$$

The inverse of $\mathbf{T}$ is expressed in terms of the generators (12) provided we have invertability of $\det_\xi T$, and $(g - \omega T^{-1}\psi)$

$$\mathbf{T}^{-1} = \begin{pmatrix} I_2 & -T^{-1}\psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & (g-\theta)^{-1} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ -\omega T^{-1} & 1 \end{pmatrix}. \tag{19}$$

Thus we arrive at the following definition.

Definition. The dual to the Hopf superalgebra $osp_\xi(1|2)$ generated by the entries of $\mathbf{T}$ (12) subject to the relations (14)–(18) is said to be the quantum supergroup $OSp_\xi(1|2)$.

Another way to define this $OSp_\xi(1|2)$ is to use the twist element $\mathcal{F}$ as the pseudodifferential operator on the Lie supergroup $OSp(1|2)$, and redefine super-commutative product of functions on this supergroup.

The reduction or Hopf superalgebra homomorphism, of $OSp_\xi(1|2)$ to $(sB_-)'_\xi$ is given by

$$b = \alpha = \beta = 0 \qquad g = 1 \qquad a = d^{-1} = \exp(\nu) \qquad \gamma a^{-1} = \delta = \tfrac{1}{2}\eta \qquad c = 2\xi x a.$$

3. Conclusion

The rank-one orthosymplectic superalgebra has been deformed by the twist element $\mathcal{F} \in \mathcal{U}(sl(2))^{\otimes 2}$ obtained from the embedded Lie algebra $sl(2)$. Although the deformed Lie superalgebra is finite dimensional it can be used for further deformation of infinite-dimensional Hopf superalgebras (e.g. super-Yangians) and integrable models [14]. There are also possibilities for different contractions. Work in this direction is in progress.

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